Chapter (II)

Mathematical Logic

Chapter (III)

Mathematical Logic

The rules of mathematical logic specify methods of reasoning mathematical statements. Greek philosopher, Aristotle, was the pioneer of logical reasoning. Logical reasoning provides the theoretical base for many areas of mathematics and consequently computer science. It has many practical applications in computer science like design of computing machines, artificial intelligence, definition of data structures for programming languages etc.

2.1 Propositional Calculus

Propositional Logic is concerned with statements to which the truth values, “true” and “false”, can be assigned. The purpose is to analyze these statements either individually or in a composite manner.

Definition.

In logic, **a proposition (or a statement)** is a meaningful declarative sentence that is either true or false, but not both.

The truth value of a proposition is True "T or 1" if it is a true proposition and false "F or 0" if it is a false proposition. Letters p, q, r , ... are used to denote proposition and are called propositional variables.

* The following propositions are true

 (i) A triangle has three sides.

 (ii) 7 is odd.

 (iii) 2 divides 24.

* The following propositions are false:

 (i) $5 + 3 = 9$.

 (ii) Makkah is the capital of Saudi Arabia.

 (iii) 2 divides 7.

* The following are not proposition:

 (1) Who are you?

 Not declarative sentences

 (2) Help yourself!

 Not declarative sentence.

 (3) $u - 2 = 1$

 Neither true nor false.

 (4) $u - v = w$.

 Neither true nor false.

(5) Broccoli tastes good.

Meaningful declarative sentences, but is not proposition but rather matters of opinion or taste.

Definition.

A formula (or a compound proposition) A formula is formed from existing propositions using connectives.

Definition.

Since we need to know the truth value of a proposition in all possible scenarios, we consider all the possible combinations of the propositions which are joined together by Logical Connectives to form the given compound proposition. This compilation of all possible scenarios in a tabular format is called a **truth table**.

In particular, truth tables can be used to tell whether a propositional expression is true or false for all legitimate input values. Practically, a truth table is composed of one column for each input variable (for example, p and q), and one final column for all of the possible results of the logical operation that the table is meant to represent (for example, $p \rightarrow q$). Each row of the truth table therefore contains one possible configuration of the input variables (for instance, p is true (written 1 or T) q is false (written 0 or F)), and the result of the operation for those values.

♣ Logical Connectives

Connectives are either unary operations like logical identity and logical negation, or binary operations like logical conjunction, logical disjunction and logical implication.

Definition. (Logical identity and logical Negation).

Let p be a proposition.

● Logical identity

Logical identity is an operation on one logical value, typically the value of a proposition that produces a value of *true* if its operand is true and a value of *false* if its operand is false. The truth table for the logical identity operator is as follows:

Logical Identity

p	p
Operand	*Value*
1	1
0	0

● Logical negation

Logical negation is an operation on one logical value, typically the value of a proposition, which produces a value of *true* if its operand is false and a value of *false* if its operand is true.

The truth table for logical negation (written as $\neg p$ or $\sim p$) is as follows:

Logical negation

p	$\neg p$
1	0
0	1

Example.

The negation of the proposition "The sun shines on the screen" is "The sun does not shine on the screen".■

We will now introduce the logical connectives (binary operations) that are used to form formulas.

Definition. (Logical Conjunction" ∧ ")

Logical conjunction is an operation on two logical values, typically the values of two propositions, that produces a value of *true* if both of its operands are true.

The truth table for p AND q (written as $p \wedge q$) is as follows:

Logical Conjunction

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

Example.

Let p be the proposition "It is sunny today" and q be the proposition "The sun shines on the screen". Then the conjunction of these propositions, $p \wedge q$, is the proposition "It is sunny today and the sun shines on the screen". This proposition is true when the day is sunny and the sun shines on the screen. It is false otherwise. ■

Definition. (Logical Disjunction" ∨ ")

Logical disjunction is an operation on two logical values, typically the values of two propositions, that produces a value of *true* if at least one of its operands is true.

The truth table for p OR q (written as $p \vee q$) is as follows:

Logical disjunction

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

Example.

The disjunction of the propositions p and q where p and q are the same propositions as in the above example, $p \vee q$, is the proposition "It is sunny today or the sun shines on the screen". This proposition is true on any day that is either sunny day or the sun shines on the screen (including both). It is only false on days that are not sunny and when it also does not shine on the screen. ■

Definition.

("Logical Implication" or "Conditional Statement" " → ")

Logical implication is associated with an operation on two logical values, typically the values of two propositions, that produces a value of *false* just in the singular case the first operand is true and the second operand is false.

The truth table associated with the Logical implication if p then q (symbolized as $p \rightarrow q$) is as

Logical implication

p	q	$p \longrightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

It may also be useful to note that $p \rightarrow q$ and $\neg p \vee q$ have the same truth table. A variety of terminology is used to express $p \rightarrow q$. Some of them are: "if *p*, then *q*", "*p* implies *q*", "if *p*, *q*" , "*p* only if *q*", "*p* is sufficient for *q*", "a sufficient condition for *q* is *p*", "*q* if *p*", "*q* whenever *p*", "*q* when *p*" , "*q* is necessary for *p*"
"a necessary condition for *p* is *q*" , "*q* follows from *p*"
and "q unless $\neg p$.

Example.

Let p the proposition "Aly study well" and q the proposition "Aly will be a Computer Science student". Then the formula $p \rightarrow q$ -as a formula in English- is "If Aly study well, then he will be a Computer Science student". ■

Definition. (Converse, Contra-positive and Inverse)

There are some related conditional statements that can be formed from $p \to q$. The conditional statement $q \to p$ is called the **converse** of $p \to q$. The **contra-positive** of $p \to q$ is the conditional statement $\neg q \to \neg p$.

The statement $\neg p \to \neg q$ is called the **inverse** of $p \to q$.

The contra-positive, $\neg q \to \neg p$, of a conditional statement $p \to q$ has the same truth value as $p \to q$.

On the other hand, neither the converse, $q \to p$, nor the inverse $\neg p \to \neg q$, has the same truth value as $p \to q$ for all possible truth values of p and q.

Example.

What are the contra-positive, the converse, and the inverse of the conditional statement "The home team wins whenever it is raining".

Solution.

Because "q whenever p" is one of the ways to express the conditional statement $p \to q$, the original statement can be rewritten as "If it is raining, then the home team wins".

Consequently, the contra-positive of this conditional statement is “If the home team does not win, then it is not raining”.

The converse is “If the home team wins, then it is raining”.

The inverse “If it is not raining, then the home team does not win”. Only the contrapositive is equivalent to the original statement. ■

We now introduce another way to combine propositions.

Definition. (Biconditional " $\leftrightarrow$ ").

Biconditional (also known as **logical equality**) is an operation on two logical values, typically the values of two propositions, that produces a value of *true* if both operands are false or both operands are true.

The truth table for p XNOR q (written as $p \leftrightarrow q$) is as follows:

Logical Equality

p	q	$p \leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

So $p \leftrightarrow q$ is true if p and q have the same truth value (both true or both false), and false if they have different truth values. There are some other ways to express $p \leftrightarrow q$ "p is necessary and sufficient for q"; "p iff q" where "iff" is the abbreviation for "if and only if" and " if p then q and conversely ".

Example.

Let p be the statement "You can pass the exam." and let q be the statement "You study well". Then $p \leftrightarrow q$ is the statement "You can pass the exam if and only if you study well". ■

Remark.

The previous operators ($\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$) are the **common operators** which we will focus on.

Definition. (Exclusive Or" $\oplus$ "**).**

Truth table for Exclusive Or " $\oplus$ "

Logical Equality

p	q	$p \oplus q$
1	1	0
1	0	1
0	1	1
0	0	0

Actually, this operator can be expressed by using other operators:

$p \oplus q$ is the same as $\neg (p \leftrightarrow q)$.

$\oplus$ is used often in CSE. So we have a symbol for it.

● Order of precedence

As a way of reducing the number of necessary parentheses, one may introduce precedence rules for operators. $\neg$ has higher precedence than $\wedge$, $\wedge$ higher than $\vee$, and $\vee$ higher than $\rightarrow$.

Here is a table that shows a commonly used precedence of logical operators.

The order of precedence determines which connective is the "main connective" when interpreting a formula.

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Example.

$\neg p \wedge q$ means $(\neg p) \wedge q$;

$p \wedge q \longrightarrow r$ means $(p \wedge q) \longrightarrow r$;

$p \vee q \wedge \neg r \rightarrow s$ is short for $\left[p \vee \left(q \wedge (\neg r)\right)\right] \rightarrow s$.

When in doubt, use parenthesis. ■

Example.

Find the truth table for the following formula: "If you studied discrete Mathematics well and did not neglect studying logic, you would gain high marks in the exam".

Solution.

Suppose that

p: studied discrete Mathematics well;

q: neglect studying logic;

r: gain high mark in the exam.

The formula is $p \wedge \neg q \longrightarrow r$

p	q	r	$\neg q$	$p \wedge \neg q$	$p \wedge \neg q \longrightarrow r$
1	1	1	0	0	1
1	1	0	0	0	1
1	0	1	1	1	1
1	0	0	1	1	0
0	1	1	0	0	1
0	1	0	0	0	1
0	0	1	1	0	1
0	0	0	1	0	1

■

● Tautologies and Contradictions

Definition.

A formula that is always true, no matter what the truth values of the propositions that occur in it, is called a **tautology**.

A formula that is always false is called **contradiction**.

A formula that is neither a tautology nor a contradiction is called a **contingency**.

Example.

We can construct examples of tautologies and contradictions using just one proposition. Consider the truth tables of $p \vee \neg p$ and $p \wedge \neg p$. Since $p \vee \neg p$ is always true, it is a tautology. Since $p \wedge \neg p$ is always false, it is a contradiction.

Example of a tautology and a contradiction

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
1	0	1	0
0	1	1	0

■

• Logical Equivalence

Definition.

Two formulas p and q are **logically equivalent**, denoted by $p \equiv q$, if and only if they have the same truth values for all possible combination of truth values for the propositional variables. Also,

Definition.

Two formulas p and q are called **logically equivalent** if $p \leftrightarrow q$ is a tautology.

Checking logical equivalence
1. Construct and compare truth tables (most powerful) 2. Use logical equivalence laws

Example.

The formulas $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

p	q	$\neg p$	$p \rightarrow q$	$\neg p \vee q$
1	1	0	1	1
1	0	0	0	0
0	1	1	1	1
0	0	1	1	1

■

Example.

The formulas $\neg(p \lor q)$ and $\neg p \land \neg q$ are logically equivalent.

p	q	$\neg$p	$\neg q$	$p \lor q$	$\neg(p \lor q)$	$\neg p \land \neg q$
1	1	0	0	1	0	0
1	0	0	1	1	0	0
0	1	1	0	1	0	0
0	0	1	1	0	1	1

Since the truth values of the formulas $\neg(p \lor q)$ and $\neg p \land \neg q$ agree for all possible combinations of the truth values of p and q, it follows that $\neg(p \lor q) \leftrightarrow \neg p \land \neg q$ is a tautology and these formulas are logically equivalent. Similarly, we can prove that $\neg(p \land q) \equiv \neg p \lor \neg q$.■

Theorem. (Algebraic properties of connectives)

(1) Commutative rules:

(a) $p \wedge q \equiv q \wedge p$, (b) $p \vee q \equiv q \vee$ p.

(2) Associative rules:

(a) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$,

(b) $(p \vee q) \vee r \equiv p \vee (q \vee r)$.

(3) Distributive rules:

(a) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$,

(b) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.

(4) Identity rules:

(a) $p \vee 0 \equiv p$, (b) $p \wedge 1 \equiv p$

(5) Negation rules:

$p \wedge \neg p \equiv 0$ and $p \vee \neg p \equiv 1$.

(6) Double negation rule:

$\neg(\neg p) \equiv p$.

(7) Idempotent rules:

$p \vee p \equiv p$ and $p \wedge p \equiv p$.

(8) De Morgan's rules:

(a) $\neg(p \wedge q) \equiv \neg p \vee \neg q$,

(b) $\neg(p \vee q) \equiv \neg p \wedge \neg q$.

(9) Universal rules:

$p \wedge 0 \equiv 0$ and $p \vee 1 = 1.$

(10) Absorption rules:

(a) $p \vee (p \wedge q) \equiv p,$

(b) $p \wedge (p \vee q) \equiv p.$

(11) Alternative proof rule:

(a) $p \longrightarrow (q \vee r) \equiv (p \wedge \neg q) \longrightarrow r \equiv (p \wedge \neg r) \longrightarrow q.$

(b) $p \vee q \longrightarrow r \equiv (p \longrightarrow r) \wedge (q \longrightarrow r).$

(12) Conditional rules:

(a) $p \longrightarrow q \equiv \neg p \vee q$

(b) $\neg(p \longrightarrow q) \equiv p \wedge \neg q.$

(13) Biconditional rules:

(a) $p \leftrightarrow q \equiv (p \longrightarrow q) \wedge (q \longrightarrow p)$

(b) $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

(c) $p \leftrightarrow q \equiv (\neg p \vee q) \wedge (p \vee \neg \mathrm{q})$

(14) Rules of contrapositive:

$$p \longrightarrow q \equiv \neg\, q \longrightarrow \neg p$$

(15) Exportation – importation rule:

$$p \longrightarrow (q \longrightarrow r) \equiv p \wedge q \longrightarrow r$$

Proof. Exercise. ◄

Example.

Use the algebraic properties of connectives to prove:

(a) $\neg\big(p \wedge (\neg p \vee q)\big) \equiv \neg p \vee \neg q$;

(b) $[(p \vee q) \wedge (p \longrightarrow r) \wedge (q \longrightarrow r)] \longrightarrow r$ is a tautology.

Solution.

(a) Exercise.

(b) $[(p \vee \mathrm{q}) \wedge \big((p \longrightarrow r) \wedge (q \longrightarrow r)\big)] \longrightarrow r$

$\equiv [(p \vee q) \wedge \big((p \vee q) \longrightarrow r\big)] \rightarrow r$

Alternative proof rule

$\equiv [(p \vee \mathrm{q}) \wedge (\neg(p \vee q) \vee \mathrm{r})] \longrightarrow r$

Conditional rule

$\equiv [((p \vee q) \wedge (\neg(p \vee q))) \vee \big((p \vee q) \wedge r\big)] \longrightarrow r$

Distributive rule

$\equiv [0 \vee ((p \vee q) \wedge r)] \longrightarrow r$ Negation rule

$\equiv [(p \vee q) \wedge r] \longrightarrow r$ Identity rule

$\equiv \neg[(p \vee q) \wedge r] \vee r$ Conditional rule

$\equiv [\neg(p \vee q) \vee \neg r] \vee r$ De Morgan's rule

$\equiv \neg(p \vee q) \vee [\neg \mathrm{r} \vee r]$ Associative rule

$\equiv \neg(p \vee q) \vee 1$ Negation rule

$\equiv 1$ Idempotent rules. ■

Exercise Set (2.1)

1- Which of the following are propositions?

(a) Buy Premium Bonds!

(b) The Apple Macintosh is a 16-bit computer.

(c) There is a largest even number.

(d) Why are we here?

(e) $8 + 7 = 13$.

(f) $a + b = 13$.

2- p is "1024 bytes is known as 1MB" and q is "A computer keyboard is an example of a data input device". Express the following formulas as English sentences in as natural a way as you can. Are the resulting propositions true or false?

(a) $p \wedge q$; (b)) $p \vee q$; ; (c) $\neg p$.

3- p is "$x < 50$"; q is "$x > 40$".

Write as simply as you can:

(a) $\neg p$; (b) $\neg q$; (c) $p \wedge q$; (d) $p \vee q$; (e) $\neg p \wedge q$;

(f) $\neg p \wedge \neg q$.

One of these compound propositional functions always produces the output *true*, and one always outputs *false*. Which ones?

4- p is "I like Math" and q is "I am going to spend at least 6 hours a week on Math". Write in as simple English as you can:

(a) $(\neg p) \wedge q$; (b) $(\neg p) \vee q$;

(c) $\neg(\neg p)$; (d) $(\neg p) \vee (\neg q)$;

(e) $\neg(p \vee q)$; (f) $(\neg p) \wedge (\neg q)$;

(g) $p \rightarrow q$; (h) $p \wedge q$.

5- Construct a truth table for each of these formulas:

(a) $p \wedge \neg p$;

(b) $p \vee \neg p$;

(c) $(p \vee \neg q) \rightarrow q$;

(d) $(p \vee q) \rightarrow (p \wedge q)$;

(e) $p \rightarrow \neg p$;

(f) $p \leftrightarrow \neg p$.

6- Show that each of these implications is a tautology by using truth tables.

(a) $[\sim p \wedge (p \vee q)] \rightarrow q$.

(b) $[(p \rightarrow q) \wedge (q \rightarrow r)] \wedge (p \rightarrow r)$

7- Show that each implication in Exercise 6 is a tautology without using truth tables.

8- Show that every pair in the following are logically equivalent:

(a) $p \to q$ and $\neg q \to \neg p$

(b) $\neg p \leftrightarrow q$ and $p \leftrightarrow \neg q$

(c) $\neg(p \leftrightarrow q)$ and $\neg p \leftrightarrow \neg q$

(d) $(p \to q) \wedge (p \to r)$ and $p \to (q \wedge r)$

(e) $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$

9- Show that $(p \vee q) \wedge (\neg p \vee r) \to (q \vee r)$ is a tautology.

10- Show that $(p \to q) \to r$ and $p \to (q \to r)$ are not logically equivalent.

11-Prove that:

(a) $p \longrightarrow q \equiv \neg\, q \longrightarrow \neg p$;

(b) $\neg(p \vee q) \equiv \neg p \vee \neg q$;

(c) $p \longrightarrow q \equiv \neg p \vee q$;

(d) $(p \wedge q) \longrightarrow r \equiv \neg r \longrightarrow (\neg p \vee \neg q)$.

2.2 Predicates and Quantifiers

(A) Predicates

Predicates are statements involving variables (called predicate variables), such as:

"$x > 3$", "$x = y + 3$", "$x + y = z$".

They are not propositions because the truth value you give them will depend on the values assigned to the variables x and y. **The domain** of a predicate variable is the set of all values that may be substituted in place of the variable.

In English you may have statements like this:

1- She is Tall and Fair.

2- x was born in a city y in the year z.

Often pronouns (I, he, she, you etc.) are used in place of variables.

In the first case - we cannot say if the statement is true because that depends of who she is and in the second case the statement will get a truth value depending on variable x, y and z.

Predicate are noted something like this $P(x, y, z)$.

For example

$P(x, y, z)$. This stands for the predicate "$x + y = z$".

$M(x, y)$. This stands for "x is married to y".

In general, you have predicates in the form of:

$P(x)$ - this is a unary predicate (has one variable).

$P(x, y)$ - this is a binary predicate (has two variables).

$P(x_1, x_2, \ldots, x_n)$ - this is an n-ray or n-place predicate – (has n individual variables in a predicate).

You have to choose the values for the variables - these can be from a set of humans - a specific human, a set of places or a place, a set of integers or an integer, a set of real numbers or a real number and so on.

The values are chosen from a particular domain of values called **a universe or a universe of discourse**.

If we take a look at this again:

x was born in a city *y* in the year *z*. *x* is taken from a set of human beings, *y* is taken from a set of cities and *z* is taken from a set of years. This is called the underlying universe.

Looking at this again:

$P(x, y, z)$.The values for the variables *x*, *y* and *z* will be taken from a set of integers or negative integers.

In some cases, you will have to specify the underlying universe because a certain predicate may be true for real numbers but false for not real numbers.

In the case *x* has to be a human being and *y* has to be a city and *z* has to be a year. You cannot have *y* as an integer or *z* a colour for example.

If you assign a particular value to each of the n place values in $P(x_1, x_2, \ldots, x_n)$ then the **predicate** becomes a **proposition** and takes a truth value - true or false.

Again *the statement "x is greater than* 3*" has two parts. The first part, the variable x, is the subject of the statement. The second part, the predicate, "is greater than* 3*", refers to a property that the subject of the statement can have. We can denote the statement "x is greater than* 3*" by* $P(x)$ *where P denotes the predicate "is greater than* 3*" and x is the variable. Once a value has been assigned to the variable x, the statement* $P(x)$ *becomes a proposition and has a truth value.*

Example.

Let $P(x)$ denote the statement "$x > 3$". What are the truth values of the propositions $P(4)$ and $P(2)$?

Solution.

We obtain the proposition $P(4)$ by setting $x = 4$ in the statement "$x > 3$". Hence $P(4)$, which is the proposition "$4 > 3$" is true.

However, $P(2)$ which is the proposition "$2 > 3$", is false.■

Example.

Let $Q(x, y)$ denote the statement "$x = y + 3$." What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?

Solution.

To obtain proposition $Q(1,2)$, set $x = 1$ and $y = 2$ in the statement $Q(x, y)$. Hence $Q(1, 2)$ is the proposition "$1 = 2 + 3$" which is false.

The proposition $Q(3, 0)$ is the proposition "$3 = 0 + 3$" which is true. ■

Example.

What are the truth values of the propositions $P(1, 2, 3)$ and $P(0, 0, 1)$, where $P(x, y, z)$ denote the statement "$x + y = z$"?

Solution.

The proposition $P(1, 2, 3)$ is obtained by setting $x = 1, y = 2$, and $z = 3$ in the statement $P(x, y, z)$. We see that $P(1, 2, 3)$ is the proposition"$1 + 2 = 3$", which is true.

Also, note that $P(0, 0, 1)$, which is the proposition"$0 + 0 = 1$" is false.■

Definition.

If $P(x)$ is a predicate and *x* has domain *D*, the **truth set** of $P(x)$ is the set of all elements of *D* that make $P(x)$ true when they are substituted for *x*. The truth set of $P(x)$ is denoted $\{x \in D : P(x)\}$ and we read as "the set of all *x* in *D* such that $P(x)$."

Example.

Let $Q(n)$ be the predicate "n is a factor of 8." Find the truth set of $Q(n)$ if:

(a) the domain of n is $\mathbb{Z}^+$, the set of all positive integers.

(b) the domain of n is $\mathbb{Z}$, the set of all integers.

Solution.

(a) The truth set is $\{1, 2, 4, 8\}$ because these are exactly the positive integers that divide 8 evenly.

(b) The truth set is $\{1,\ 2,\ 4,\ 8, -1, -2, -4, -8\}$ because the negative integers $-1, -2, -4$, and -8 also divide into 8 without leaving a remainder. ■

Definition.

Let $P(x)$ and $Q(x)$ be predicates with common domain D of x. The notation $P(x) \Rightarrow Q(x)$ means that every element in the truth set of $P(x)$ is in the truth set of $Q(x)$. Similarly, $P(x) \Leftrightarrow Q(x)$ means that $P(x)$ and $Q(x)$ have identical truth sets.

Example.

Let $P(x)$ be "x is a factor of 8",

$Q(x)$ be "x is a factor of 4",

$R(x)$ be " $x < 5$ and $x \neq 3$",

and let the domain of x be set of positive integers. Then

Truth set of $P(x)$ is $\{1, 2, 4, 8\}$.

Truth set of $Q(x)$ is $\{1, 2, 4\}$.

Since every element in the truth set of $Q(x)$ is in the truth set of $P(x)$, then $Q(x) \Rightarrow P(x)$.

Further, truth set of $R(x)$ is $\{1, 2, 4\}$, which is identical to the truth set of $Q(x)$. Hence $R(x) \Leftrightarrow Q(x)$. ■

(B) Quantifiers

(i) The Universal Quantifier " ∀ "

One sure way to change predicates into propositions is to assign specific values to all their variables.

For example, if x represents the number 35, the sentence "x is divisible by 5" is a true proposition.

Another way to obtain propositions from predicates is to add quantifiers. Quantifiers are words that refer to quantities such as "some" or "all" and tell for how many elements a given predicate is true.

The symbol ∀ is called the universal quantifier.

Depending on the context, it is read as "for every," "for each," "for any," "given any," or "for all."

For example, another way to express the sentence

"Every human being is mortal"

or

"All human beings are mortal"

is to write

"∀ human beings x, x is mortal",

which you would read as

"For every human being x, x is mortal."

If you let D be the set of all human beings, then you can symbolize the statement more formally by writing

"$\forall x \in D$, x is mortal".

In sentences containing a mixture of symbols and words, the $\forall$ symbol can refer to two or more variables.

For instance, you could symbolize

"For all real numbers x and y, $x + y = y + x$."

as

"$\forall$ real numbers x and y, $x + y = y + x$."

Definition.

Let $P(x)$ be a predicate and D the domain of x. A universal quantification of $P(x)$ is a proposition of the form "$\forall x \in D, P(x)$." It is defined to be true if, and only if, $P(x)$ is true for each individual x in D. It is defined to be false if, and only if, $P(x)$ is false for at least one x in D.

The notation $\forall x P(x)$ is used for the universal quantification of $P(x)$ when the domain is known.

Here $\forall$ is called the **universal quantifier**.

Example.

Let $P(x)$ be the statement "$x + 1 > x$". What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?

Solution.

Since $P(x)$ is true for all real numbers x, the quantification $\forall x P(x)$ is true. ■

Example.

Let $Q(x)$ be the statement "$x < 2$". What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?

Solution.

$Q(x)$ is not true for every real number x, since, for instance, $Q(3)$ is false. Thus $\forall x Q(x)$is false. ■

Note.

When all the elements in the universe of discourse can be listed, say $x_1, x_2, \dots, x_n$ it follows that the universal quantification $\forall x P(x)$ is the same as the conjunction $P(x_1) \wedge P(x_2) \wedge \dots \wedge P(x_n)$.

Example.

What is the truth value of $\forall x P(x)$, where $P(x)$ is the statement "$x^2 < 10$" and the universe of discourse consists of positive integers not exceeding 4?

Solution.

The statement $\forall x P(x)$ is the same as the conjunction $P(1) \wedge P(2) \wedge P(3) \wedge P(4)$. Since $P(4)$,which is the statement"$4^2 < 10$" , is false, so $\forall x P(x)$ is false. ■

To show that a statement of the form $\forall x P(x)$ is false, where $P(x)$ is a propositional function, we need only find one value of x in the universe of discourse for which $P(x)$ is false. Such a value of x is called a **counterexample** to the statement $\forall x P(x)$.

Example.

Suppose that $P(x)$ is "$x^2 > 0$". To show the statement $\forall x P(x)$ is false where the universe of discourse consists of all integers, we give a counterexample. We see that $x = 0$ is a counterexample since $x^2 = 0$ when $x = 0$ so that x^2 is not greater than 0 when $x = 0$. ■

(ii) The Existential Quantifier " ∃ "

The symbol ∃ denotes "there exists" and is said to be the existential quantifier. For example, the sentence

"There is a student in Math211"

can be written as

"∃ a person *x* such that *x* is a student in Math211",

or, more formally,

"$\exists x \in P$ such that *x* is a student in Math211",

where P is the set of all people.

The domain of the predicate variable is generally indicated either between the ∃ symbol and the variable name or immediately following the variable name, and the words such that are inserted just before the predicate. Some other expressions that can be used in place of there exists are there is a, we can find a, there is at least one, for some, and for at least one.

In a sentence such as

"∃ integers *m* and *n* such that $m + n = m \cdot n$,"

the ∃ symbol is understood to refer to both *m* and *n*.

In more formal versions of symbolic logic, the words such that are not written out (although they are understood) and a separate $\exists$ symbol is used for each variable: "$\exists m \in \mathbb{Z}\,(\exists n \in \mathbb{Z}(m+n = m \cdot n))$."

Definition.

Let $P(x)$ be a predicate and D the domain of x. An existential statement is a statement of the form

"$\exists x \in D$ such that $P(x)$."

It is defined to be true if, and only if, $P(x)$ is true for at least one x in D. It is false if, and only if, $P(x)$ is false for all x in D.

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$.

Here $\exists$ is called the **existential quantifier**.

A domain must always be specified when a statement $\exists x P(x)$ is used. Furthermore, the meaning of $\exists x P(x)$ changes when the domain changes. Without specifying the domain, the statement $\exists x P(x)$ has no meaning. The existential quantification $\exists x P(x)$ is read as:

"There is an x such that $P(x)$","There is at least one x such that $P(x)$" or "For some x $P(x)$".

Example.

Let $P(x)$ denote the statement "$x > 3$". What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real numbers?

Solution.

Because "$x > 3$" is sometimes true - for instance, when $x = 4$, the existential quantification $\exists x P(x)$ of $P(x)$ is true. ■

Example.

Let $Q(x)$ denote the statement "$x = x + 1$". What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real numbers?

Solution.

Because $Q(x)$ is false for every real number x, the existential quantification of $Q(x)$ which is $\exists x P(x)$ is false. ■

When all elements in the domain can be listed say $x_1, x_2, \dots, x_n$ the existential quantification $\exists x P(x)$ is the same as the disjunction $P(x_1) \vee P(x_2) \vee \dots \vee P(x_n)$

because this disjunction is true if and only if at least $P(x_1), P(x_2), \ldots, P(x_n)$ is true.

Example.

What is the truth value of $\exists x P(x)$, where $P(x)$ is the statement "$x^2 > 10$" and the domain consists of the positive integers not exceeding 4?

Solution.

As the domain is $\{1, 2, 3, 4\}$, the proposition $\exists x P(x)$ is the disjunction $P(1) \vee P(2) \vee P(3) \vee P(4)$.

Because $P(4)$, which is the statement "$4^2 > 10$", is true, it follows that $\exists x P(x)$ is true. ■

●Translating from Formal to Informal Language

Example.

Rewrite the following formal statements in a variety of equivalent but more informal ways. Do not use the symbol ∀ or ∃.

(a) $\forall x \in \mathbb{R}, x^2 \geq 0$;

(b) $\forall x \in \mathbb{R}, x^2 \neq -1$;

(c) $\exists m \in \mathbb{Z}$ such that $m^2 = m$.

Solution.

(a) Every real number has a nonnegative square.

Or: All real numbers have nonnegative squares.

Or: Any real number has a nonnegative square.

Or: The square of each real number is nonnegative.

(b) All real numbers have squares that do not equal -1.

Or: No real numbers have squares equal to -1.

(The words none are or no … are equivalent to the words all are not.)

(c) There is a positive integer whose square is equal to itself.

Or: We can find at least one positive integer equal to its own square.

Or: Some positive integer equals its own square.

Or: Some positive integers equal their own squares. ■

Another way to restate universal and existential statements informally is to place the quantification at the end of the sentence. For instance, instead of saying "For any real number x, x^2 is nonnegative," you could say "x^2 is nonnegative for any real number x." In such a case the quantifier is said to "**trail**" the rest of the sentence.

●Trailing Quantifiers

Example.

Rewrite the following statements so that the quantifier trails the rest of the sentence.

(a) For any integer n, $2n$ is even.

(b) There exists at least one real number x such that $x^2 \leq 0$.

Solution.

(a) $2n$ is even for any integer n.

(b) $x^2 \leq 0$ for some real number x.

Or: $x^2 \leq 0$ for at least one real number x. ■

●Translating from Informal to Formal Language

Example.

Rewrite each of the following statements formally. Use quantifiers and variables.

(a) All triangles have three sides.

(b) No dogs have wings.

(c) Some programs are structured.

Solution.

(a) ∀ triangle t, t has three sides.

Or: $\forall t \in T$, t has three sides (where T is the set of all triangles).

(b) ∀ dog d, d does not have wings.

Or: $\forall d \in D$, d does not have wings (where D is the set of all dogs).

(c) ∃ a program p such that p is structured.

Or: $\exists p \in P$ such that p is structured (where P is the set of all programs). ■

●Universal Conditional Statements

A reasonable argument can be made that the most important form of statement in mathematics is the **universal conditional statement**:

$$\forall x, \text{ if } P(x) \text{ then } Q(x).$$

Familiarity with statements of this form is essential if you are to learn to speak mathematics.

●Writing Universal Conditional Statements Informally

Example.

Rewrite the following statement informally, without quantifiers or variables.

$\forall x \in \mathbb{R}$, if $x > 2$, then $x^2 > 4$.

Solution.

If a real number is greater than 2, then its square is greater than 4.

Or: Whenever a real number is greater than 2, its square is greater than 4.

Or: The square of any real number greater than 2 is greater than 4.

Or: The squares of all real numbers greater than 2 are greater than 4. ■

Example.

Rewrite each of the following statements in the form

∀ ... ., if……., then……… .

(a) If a real number is an integer, then it is a rational number.

(b) All bytes have eight bits.

(c) No fire trucks are green.

Solution.

(a) ∀ real number *x*, if *x* is an integer, then *x* is a rational number.

Or: $\forall x \in \mathbb{R}$, if $x \in \mathbb{Z}$ then $x \in \mathbb{Q}$.

(b) $\forall x$, if *x* is a byte, then *x* has eight bits.

(c) $\forall x$, if *x* is a fire truck, then *x* is not green. ■

●Equivalent Forms of Universal and Existential Statements

Observe that the two statements

"∀ real number x, if x is an integer then x is rational"

and

"∀ integer x, x is rational"

mean the same thing because the set of integers is a subset of the set of real numbers. Both have informal translations

"All integers are rational."

In fact, a statement of the form

$$\forall x \in U, \text{ if } P(x) \text{ then } Q(x)$$

can always be rewritten in the form

$$\forall x \in D, Q(x)$$

by narrowing U to be the subset D consisting of all values of the variable x that make $P(x)$ true. Conversely, a statement of the form

$$\forall x \in D, Q(x)$$

can be rewritten a

$$\forall x, \text{ if } x \text{ is in } D \text{ then } Q(x)$$

Example.

Rewrite the following statement in the two forms

"$\forall x$, if…….. then………. "

and

"$\forall$ ……… x,………. ":

"All squares are rectangles" .

Solution.

"$\forall x$, if x is a square then x is a rectangle".

and

"$\forall$ square x, x is a rectangle". ■

Similarly, a statement of the form

"$\exists x$ such that $P(x)$ and $Q(x)$"

can be rewritten as

"$\exists x \in D$ such that $Q(x)$,"

where D is the set of all x for which $P(x)$ is true.

Example.

A **prime number** is an integer greater than 1 whose only positive integer factors are itself and 1.

Consider the statement

"There is an integer that is both prime and even."

Let $P(n)$ be "n is prime" and $E(n)$ be "n is even."

Use the notation $P(n)$ and $E(n)$ to rewrite this statement in the following two forms:

a. $\exists n$ such that … … … ∧ … … ….

b. ∃ … … … n such that ……….

Solution.

(a) $\exists n$ such that $P(n) \wedge E(n)$.

(b) Two answers:

∃ a prime number *n* such that $E(n)$.

∃ an even number *n* such that $P(n)$. ■

Example.

What do the following statements mean, where the domain in each case consists of the real numbers?

(1) $\forall x < 0(x^2 > 0)$;

(2) $\forall y \neq 0(y^3 \neq 0)$;

(3) and $\exists z > 0(z^2 = 2)$.

Solution.

(1) The statement $\forall x < 0(x^2 > 0)$ states that for every real number *x* with $x < 0$, $x^2 > 0$. That is, it states "The square of a negative real number is positive".

This statement is the same as $\forall x\big(x < 0 \to (x^2 > 0)\big)$.

(2) The statement $\forall y \neq 0\ (y^3 \neq 0)$, states that for every real number y with $y \neq 0$, we have $y^3 \neq 0$ that is, it states

"the cube of every nonzero real number is nonzero."

Note that this statement is equivalent to

$$\forall y(y \neq 0 \longrightarrow y^3 \neq 0).$$

(3) The statement $\exists z > 0(z^2 = 2)$ states that there exists a real number z with $z > 0$ such that $z^2 = 2$. That is, it states

"there is a positive root of 2."

This statement is equivalent to $\exists z(z > 0 \land z^2 = 2)$.■

● Precedence of Quantifiers

The quantifiers ∀ and ∃ have higher precedence than all logical operators from propositional calculus. For example, $\forall x P(x) \lor Q(x)$ is the disjunction of $\forall x P(x)$ and $Q(x)$. In other words, it means $(\forall x P(x)) \lor Q(x)$ rather than $\forall x(P(x) \lor Q(x))$.

♣Logical Equivalence Involving Quantifiers

Definition.

Statements involving predicates and quantifiers are **logically equivalent** if and only if they have the same truth value no matter which predicates are substituted into these statements. We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

Example.

Show that $\forall x(P(x) \land Q(x))$ and $\forall x P(x) \land \forall x Q(x)$ are logically equivalent, where the same domain is used throughout.

Solution.

To show that these statements are logically equivalent, we must show that they always take the same truth value, no matter what predicate P and Q are, and no matter which domain of discourse is used.

Suppose we have particular predicates P and Q, with a common domain. We can show that $\forall x(P(x) \land Q(x))$ and $\forall x P(x) \land \forall x Q(x)$ are logically equivalent by doing two things. First, we show that if $\forall x(P(x) \wedge Q(x))$ is true, then $\forall x P(x) \wedge \forall x Q(x)$ is true.

Second, we show that if $\forall x P(x) \wedge \forall x Q(x)$ is true, then $\forall x(P(x) \wedge Q(x))$ is true.

So, suppose that $\forall x(P(x) \wedge Q(x))$ is true. This means that if a is in the domain, then $P(a) \wedge Q(a)$ is true. Hence $P(a)$ is true and $Q(a)$. Because $P(a)$ is true and $Q(a)$ for every element in the domain, we can conclude that $\forall x P(x)$ and $\forall x Q(x)$ are both true. This means that $\forall x P(x) \wedge \forall x Q(x)$ is true.

Next, suppose that $\forall x P(x) \wedge \forall x Q(x)$ is true. It follows that $\forall x P(x)$ is true and $\forall x Q(x)$ is true. Hence if a is in the domain, then $P(a)$ is true and $Q(a)$ is true. It follows that for all a, $P(a) \wedge Q(a)$ is true. It follows that $\forall x(P(x) \wedge Q(x))$ is true.

Therefore $\forall x\big(P(x) \wedge Q(x)\big) \equiv \forall x P(x) \wedge \forall x Q(x)$. ■

Exercise.

Prove that $\exists x\big(p(x) \vee Q(x)\big) \equiv \exists x p(x) \vee \exists x Q(x)$, where the same domain is used throughout.

Chapter (VIII)

Graph Theory

8.1 Introduction

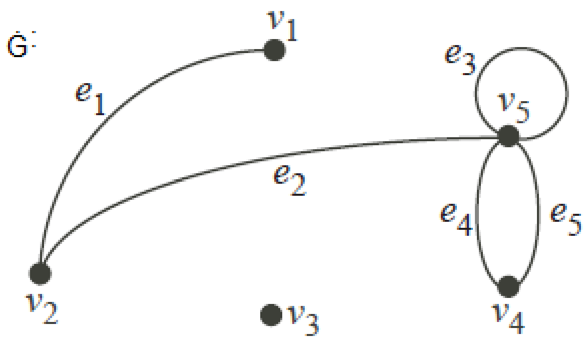
Graphs are discrete structures consisting of vertices and edges that connect these vertices. Problems in almost every conceivable discipline can be solved using graph models. Using graph models, we can determine whether it is possible to walk down all the streets in a city without going down a street twice, and we can find the number of colors needed to color the regions of a map. Graphs can be used to determine whether two computers are connected by a communications link using graph modules of computer networks. Also, graphs can be used to determine whether a circuit can be implemented on a planner circuit board. Graph with weights assigned to their edges can be used to solve problems such as finding the shortest path between two cities in a transportation network.

This chapter will introduce the basic concepts of graph theory and present many different graph models.

8.2 Graphs and Graph Models

Definition.

Conceptually, a ***graph*** is formed by ***vertices*** and ***edges*** connecting the vertices.



Formally. Let V be a non-empty set, E be another set, and f be a mapping such that $f: E \to \{\{x, y\}: x, y \in V\}$. Then the triple $G = (V, E, f)$ is called a ***graph***.

We call that V (or $V(G)$) the set of ***vertices*** of G and E (or $E(G)$) the set of ***edges*** (lines) of G. The graph $G = (V, E, f)$ is finite if each V and E is finite. We consider only the *finite graphs* without explicitly state.

☻ If $v \in f(e)$, then v is an vertex for e.

☻ If $a, b \in V$, then a is ***adjacent*** to b if there exists $e \in E$ such that $f(e) = \{a, b\}$.

☻ Also, $a \in V$ is adjacent to itself if there exists $e \in E$ such that $f(e) = \{a\}$ and e is called a ***loop*** at a.

☻ If $e_{1,} e_2 \in E$ are incident with a common vertex, then we say e_1 and e_2 ***adjacent edges***.

☻ If $f(e_1) = f(e_2) = \{a, b\}$, then e_1 and e_2 are called **a *multiple edge*.**

☻ **I**f $f(e_1) = f(e_2) = \{v\}$, then e_1 and e_2 are called **a** ***multiple loop*** at v.

☻ A graph G with no loops and no multiple edges is **a** ***simple graph***.

☻ If $G = (V, E, f)$ is a graph and $f(e) = \{a, b\}$, then we write $e = \{a, b\}$ and so we write $G = (V, E)$ instead of $G = (V, E, f)$.

We sometimes consider the following generalizations of graphs: a ***multigraph*** is a pair (V, E) where V is a set and E is a ***multiset*** of unordered pairs from V . In other words, we allow more than one edge between two vertices. A ***pseudograph*** is a pair (V, E) where V is a set and E is a ***multiset*** of unordered multisets of size two from V . A pseudograph allows ***loops***, namely edges of the form $\{a, a\}$ for $a \in V$.

☻ In general, we visualize graphs by using points to represent vertices and line segments, possibly curved, to represent edges.

Definition.

The set of all neighbors of a vertex v of $G = (V, E)$, denoted by $N(v)$, is called the neighborhood of v. If A is a subset of V , we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A. So, $N(A) = \bigcup_{v \in A} N(v)$.

To keep track of how many edges are incident to a vertex, we make the following definition.

Definition.

Let $G = (V, E)$ be a graph and $x \in V$. The **degree** of x (denoted by $d_G(x)$) is the number of edges incident with it, except a loop at x contributes twice to the degree of x.

☻ If $d_G(x) = 0$, then x is said to be **isolated** vertex.

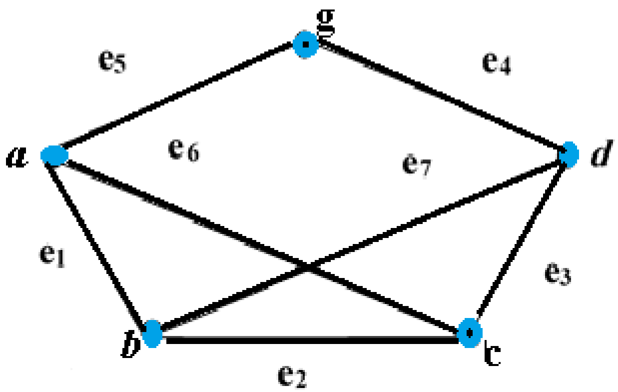
☻ A vertex is **pendant** if and only if it has degree one.

☻ A vertex with odd degree is said to be **odd vertex** and one with even degree is said to be **even vertex**.

☻ The degree sequence of a graph G is the ***sequence*** of degrees of vertices of G in non-increasing order.

Note.

We represent a graph by means of a diagram.



Graph H:

Thus, in the graph H:

☻ The points a and b are adjacent, but a and d are not.

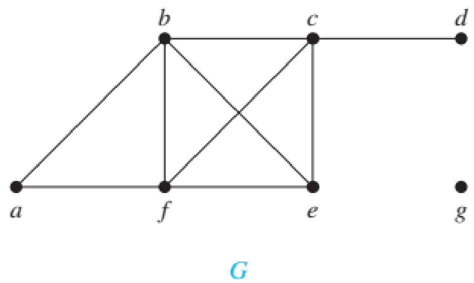
☻ The lines e_2 and e_6 are adjacent but e_6 and e_7 are not.

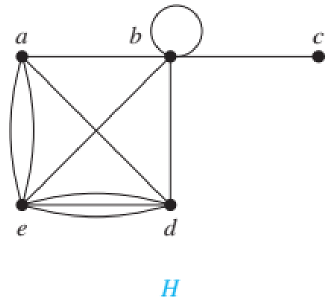
☻ Although the lines e_6 and e_7 are intersect in the diagram but their **intersection** is not a vertex of the graph.

☻ The degree sequence of the graph H is (3,3,3,3,2).

Example.

What are the degrees and what are the neighborhoods of the vertices in the graphs G and H displayed in the given figure?





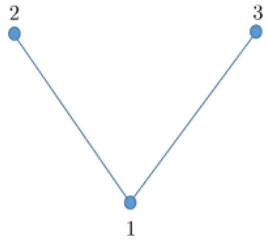
Solution.

In G, $d_G(a) = 2$, $d_G(b) = d_G(c) = d_G(f) = 4$, $d_G(d) = 1$, $d_G(e) = 3$, and $d_G(g) = 0$. The neighborhoods of these vertices are $N(a) = \{b, f\}$, $N(b) = \{a, c, e, f\}$, $N(c) = \{b, d, e, f\}$, $N(d) = \{c\}$, $N(e) = \{b, c, f\}$, $N(f) = \{a, b, c, e\}$, and $N(g) = \phi$.

In H, $d_H(a) = 4$, $d_H(b) = d_H(e) = 6$, $d_H(c) = 1$, and $d_H(d) = 5$. The neighborhoods of these vertices are $N(a) = \{b, d, e\}$, $N(b) = \{a, b, c, d, e\}$, $N(c) = \{b\}$, $N(d) = \{a, b, e\}$, and $N(e) = \{a, b, d\}$.■

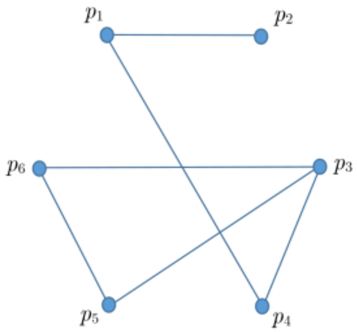
Example.

Consider the graph $G = (V, E)$, where $V = \{1, 2, 3\}$ and $E = \{\{1, 2\}, \{1, 3\}\}$. Then the given drawing represents this graph.■



Example.

Let $V = \{p_1, p_2, p_3, p_4, p_5, p_6\}$ be a set of six people at a party, and suppose that p_1 shook hands with p_2 and p_4, p_3 shook hands with p_4; p_5 and p_6, and p_5 and p_6 shook hands. Let $G = (V, E)$ be the graph with edge set E consisting of pairs of people who shook hands. Then



$E = \{\{p_1, p_2\}, \{p_1, p_4\}, \{p_3, p_4\}, \{p_3, p_5\}, \{p_3, p_6\}, \{p_5, p_6\}\}$

A drawing of G is given in given figure. ■

Example.

Let $\mathbb{Z}$ denote the set of integers and let

$V = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : 0 \leq x \leq 2, 0 \leq y \leq 2\}$:

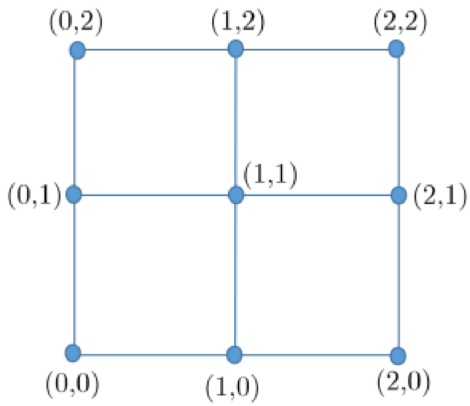
Then V is just the set of points in the plane with integer co-ordinates between 0 and 2. Now, suppose $G = (V, E)$

is the graph where E is the set of pairs of vertices of V at distance 1 from each other. In other words, (x, y) and (x', y') are adjacent iff $(x - x')^2 + (y - y')^2 = 1$.

We check that the edge set is

$E = \{\{(0,0)(0,1)\}, \{(0,0)(1,0)\}, \{(0,1)(0,2)\},$
$\{(1,0)(2,0)\}, \{(1,0)(1,1)\}, (\{1,1)(1,2)\}, \{(1,1),(2,1)\},$
$\{(0,1),(1,1)\}, \{(0,2)(1,2)\}, \{(2,0)(2,1)\}, \{(2,1),(2,2)\},$
$\{(1,2),(2,2)\}\}$:

This is a cumbersome way to write the edge set of G, as compared to the drawing of G in the given figure, which is much easier to absorb. The graph is called **grid** graph. ■



Example.

Let V be the set of binary strings of length three, so

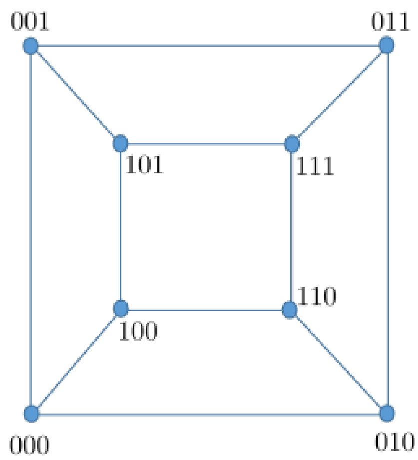
$V = \{000, 001, 010, 100, 011, 101, 110, 111\}$:

Then let E be the set of pairs of strings which differ in one position. Then

$E = \{\{000, 001\}, \{010, 000\}, \{100, 000\}, \ldots, \{111, 101\},$

$\{111, 110\}, \{111, 011\}\}$:

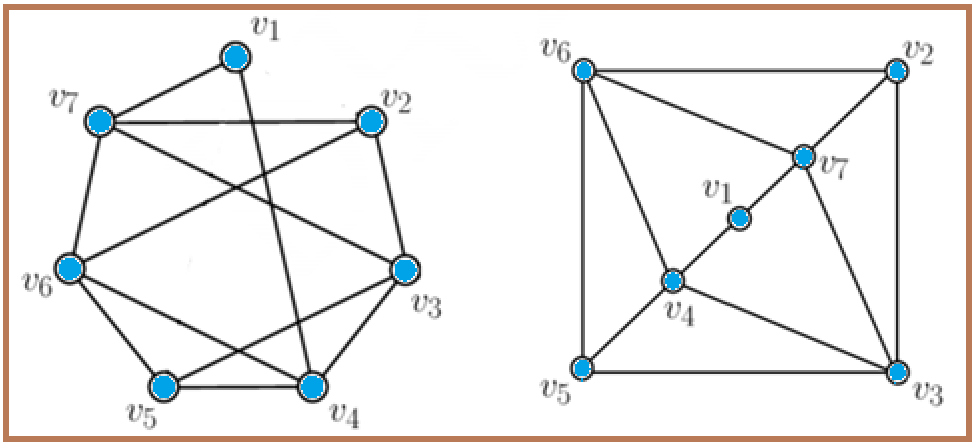
The reader should fill in the rest of the edges as an exercise. Once again, this graph actually has a very nice drawing (which explains why it is sometimes called the **cube** graph).



Example.

Consider the graph $G = (V, E)$, where the vertex set is $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and the edge set is $E = \{\{v_1, v_4\}, \{v_1, v_7\}, \{v_2, v_3\}, \{v_2, v_6\}, \{v_2, v_7\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_3, v_7\}, \{v_4, v_5\}, \{v_4, v_6\}, \{v_5, v_6\}, \{v_5, v_7\}\}$:

In the following figure, two drawings of G are shown (the reader should verify that they are both drawings of G)



Example.

Let $G = (V, E)$ be a graph, where $V = \{a, b, c, d, g\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\} = \{\{a\}, \{a, b\}, \{a, c\}, \{a, c\}, \{b, c\}, \{c, d\}\}$

1. Represent the graph G;

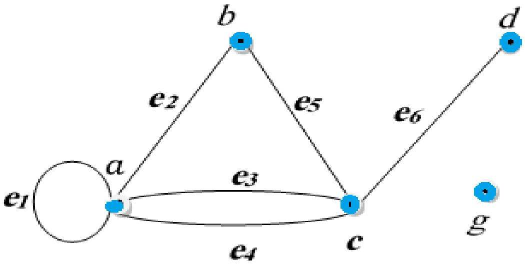
2. Find the degree of each vertex and isolated vertices;

3. Find multiple edges and loops;

4. Is G a simple graph? Why?

Solution.

1.



Graph G:

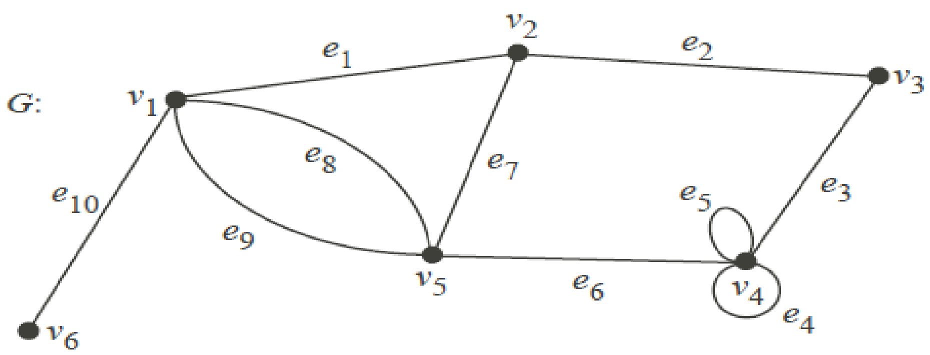
2. $d_G(a) = 5, d_G(b) = 2, d_G(c) = 4, d_G(d) = 1,$ $d_G(g) = 0$. Therefore the degree sequence is $(5, 4, 2, 1, 0)$. Since $d_G(g) = 0$ then g is the only isolated vertex.

3. Since $e_3 = e_4 = \{a, c\}$, e_3 and e_4 are multiple edges and hence G is a multiple graph. Also, since $e_1 = \{a\}$, then e_1 is a loop.

4. G is not a simple graph. It is a pseudograph as it contains multiple edges and a loop. ■

Example.

If $G = (V, E, f)$ is the graph given by the following diagram



Find V, E, f.

Solution.

It is clear that $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. and $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$.

The following table represents the function *f*:

E	e_1	e_2	e_3	e_4	e_5
$f(e)$	$\{v_1, v_2\}$	$\{v_2, v_3\}$	$\{v_3, v_4\}$	$\{v_4\}$	$\{v_4\}$

E	e_6	e_7	e_8	e_9	e_{10}
$f(e)$	$\{v_4, v_5\}$	$\{v_5, v_2\}$	$\{v_1, v_5\}$	$\{v_1, v_5\}$	$\{v_1, v_6\}$

.■

Definition.

We write $\delta(G) = \min\{d_G(v) : v \in V\}$ and $\Delta(G) = \max\{d_G(v) : v \in V\}$ for the ***minimum degree*** and ***maximum degree*** of G, respectively.

Note.

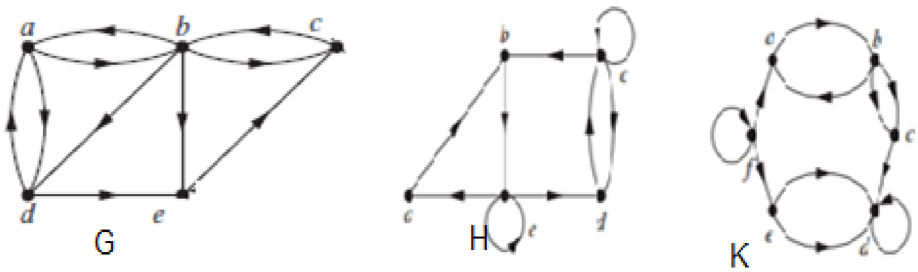
The graphs we have introduced are **undirected graphs**. Their edges are also said to be undirected. To construct a graph model, we may find it is necessary to assign direction to the edges of a graph.

Definition.

A directed graph (or digraph) $G = (V, E, f)$ consists of a non-empty set of vertices V and set of directed edges (or arcs) with the map $f: E \to \{(x, y): x, y \in V\}, i.e.$, each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to start at u and end at v. If $f(e_1) = f(e_2)$ in digraph, then e_1 and e_2 are multiple edges. If a digraph G contains no multiple edges or graph loops, then it a *directed simple graph*.

Example.

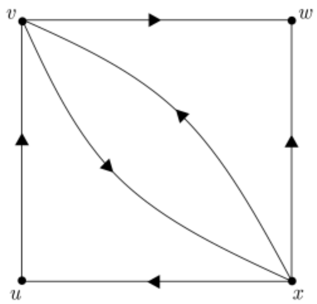
G is a simple directed graph while H and K are not.



Note:

(a) If $e = (u, v)$ is an edge of a digraph G, then u is the *initial* vertex and v is the *terminal* vertex for the edge e.

(b) In a digraph G, let $N^+(v)$ and $N^-(v)$ denote the sets of vertices adjacent from v and to v, respectively. These are the *out-neighborhood* of v and the *in-neighborhood* of v respectively. Thus $N^+(v) = \{u: (v, u) \in E\}$ and $N^-(v) = \{u: (u, v) \in E\}$. For example, in the digraph drawn below, $N^+(x) = \{u, v, w\}$ and $N^-(x) = \{v\}$.



(c) A graph with both directed and undirected edge is called a **mixed** graph.

Graph Terminology.

Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

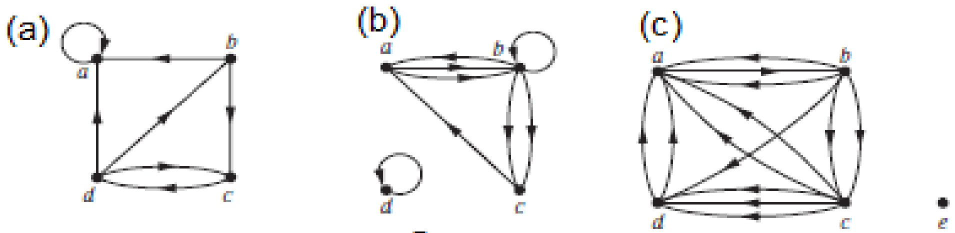
Definition.

In a graph with directed edge the **in-degree** of a vertex v, denoted by (or $d_G^-(v)$) is the number of edges with v as their terminal vertex. The **out-degree** of a vertex v denoted by (or $d_G^+(v)$)) is the number of edges with v as their initial vertex. A loop at v contributes one to the in-degree and one to the out-degree of v. In other words,

$d_G^-(v) = |N^-(v)|$ and $d_G^+(v) = |N^+(v)|$.

Example.

Find the in-degree and out-degree of each vertex in the digraph G Shown in the following diagram.



G

Solution.

The following tables gives the out-degree and in-degree of each vertex in Graphs G-(a), G-(b) and G-(c), respectively.

G-(a): G-(b):

v	a	b	c	d		a	b	c	d
$d_G^-(v)$	3	1	2	1		2	3	2	1
$d_G^+(v)$	1	2	1	3		2	4	1	1